**The Theory of Four Equidistant Rational Numbers**

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**Abstract**

*This paper serves to outline and describe a previously unexplored relationship between four “equidistant numbers”, a term that here means four numbers chosen such that each one is separated from the previous term by a constant numerical difference. The idea emerged from observing patterns within the number plates of cars, and has been generalized to encompass any 4 equidistant real numbers. The theorem states:* ***“The absolute difference between the product of the first and last term, and the second and third term of a sequence of four equidistant real numbers is always equal to two times the numerical distance squared.”*** *This paper proves the relationship for all real numbers. It was found that this relationship holds true for the set of real numbers.*

*This concept is related to Number Theory, the study of relationships of numbers. The primary method of proving the theorem is multi-dimensional induction.*

**Introduction**

Four equidistant numbers are a fairly common sight in our everyday lives. They can be seen on number plates, on digital watches or on those annoying order numbers that we receive when attempting to buy fast food at airports. But one entirely unknown fact regarding four equidistant numbers is that there in an inherent relationship among them.

The proposed theorem states that the absolute difference between the products of the first and fourth numbers, and the second and third numbers is equal to twice the numerical distance squared. This paper outlines the concept, proves the theorem in two different ways and shows that it is indeed valid for any four equidistant real numbers. The idea proposed is related to the field of number theory, and although it is not particularly mathematically intensive it proposes a novel relationship. The primary purpose of the paper is to investigate whether the proposed theorem is accurate for the described set of numbers.

Proving the theorem requires only simple algebra. There is a dearth of past information on the topic of equidistant numbers, and as a result there are no papers cited in this paper.

**Theorem**

The theorem is hypothesized to apply to all real numbers, which means that it should hold true for any and all real numbers used as the starting number and the numerical distance. We can begin with a simple example. Assume that the first number in the sequence of four is 2, and that the common difference is equal to 3. This would mean that the four numbers are:

2 5 8 11

Using these numbers we can create an equation and test the theorem. The following equation emerges:

For a starting value of 2 and a common difference of 3

2(3)2 = |(2+3)(2+6) – (2)(2+9)|

18 = |(5)(8) – (2)(11)|

18 = |40 – 22|

18 = |18|

18 = 18

This simple application validates the theorem for the numbers 2 and 3. The difference of the products was equal to twice the distance squared

**Proof 1.1**

In order to prove the theorem for any real number we must generalize the process followed in the previous two examples to any pair of real numbers. From the examples, we can extract the method used and apply it to investigate the theorem without assigning specific numbers.

In generalizing the process, we can refer to basic arithmetic sequences. In these sequences each term is obtained by adding a multiple of the common difference to the first term, each term can be expressed as the first term plus some multiple of the common difference. A set of four equidistant real numbers can also act as an arithmetic sequence.

We can represent the starting value, or first term of the sequence, by *x*, and the common numerical difference by *y*. Both of these letters represent unknown real numbers. It follows that the sequence of four numbers will look like this:

1 2 3 4

x x+y x+2y x+3y

Using this sequence of numbers and the extracted method of validating the theorem we can set up the general equation: 2*y2* = |(*x+y*)(*x+*2*y*) – (*x*)(*x+*3*y*)|

Solving this equation algebraically yields the following result:

2*y2* = |(*x+y*)(*x+*2*y*) – (*x*)(*x+*3*y*)|

2*y2* = |(*x*2+2*xy+xy+*2*y*2) – (*x*2+3*xy*)|

2*y2* = |(*x*2+3*xy+*2*y*2) – (*x*2+3*xy*)|

2*y2* = |*x*2+3*xy+*2*y*2 – *x*2–3*xy*|

2*y2* = |3*xy+*2*y*2 – 3*xy*|

2*y2* = |2*y*2|

2*y2* = 2*y*2

It is apparent that the RHS simplifies to the LHS, showing us that the theorem does indeed hold true for all real numbers. By solving the general equation the theorem has been proven.

Another method of proving this theorem is through multidimensional induction, as illustrated below.

**Proof 1.2**

To prove this theorem using multidimensional induction the first step would be to show that the equation works for the value n = 1. Or in this case, x = 1, with the constant value y. The second would be the reverse, showing that it works with y = 1 for a constant value x.

Step 1:

2*y2* = |(1*+y*)(1*+*2*y*) – (1)(1*+*3*y*)|

2*y2* = |(1+2*y*+*y*+2*y2*) – (1+3*y*)|

2*y2* = |(1+3*y+*2*y*2) – (1+3*y*)|

2*y2* = |1+3*y+*2*y*2 – 1+3*y*|

2*y2* = |3*y+*2*y*2 – 3*y*|

2*y2* = |2*y*2|

2*y2* = 2*y*2

Step 2:

2(1)2 = |(x+1)(x+2) – (x)(x+3)|

2 = |(x2+2x+x+2) – (x2+3x)|

2 = |(x2+3x+2) – (x2+3x)|

2 = |x2+3x+2 – x2+3x|

2 = |3x+2 – 3x|

2 = |2|

2 = 2

Seeing as these are both true, we move on the next step, assuming n = k, or in this case x = k for a constant value y. And y = k for a constant value x.

Step 3:

2*y2* = |(*k+y*)(*k+*2*y*) – (*k*)(*k+*3*y*)|

2*y2* = |(*k2*+2*ky*+*ky*+2*y2*) – (*k2*+*3ky*)|

2*y2* = |(*k2*+3*ky+*2*y*2) – (*k2*+3*y*)|

2*y2* = | *k2*+3*ky+*2*y*2 – *k2*–3*y*|

2*y2* = |3*ky+*2*y*2 – 3*ky*|

2*y2* = |2*y*2|

2*y2* = 2*y*2

Step 4:

2*k2* = |(*x+k*)(*x+*2*k*) – (*x*)(*x+*3*k*)|

2*k2* = |(*x*2+2*xk+xk+*2*k*2) – (*x*2+3*xk*)|

2*k2* = |(*x*2+3*xk+*2*k*2) – (*x*2+3*xk*)|

2*k2* = |*x*2+3*xk+*2*k*2 – *x*2–3*xk*|

2*k2* = |3*xk+*2*k*2 – 3*xk*|

2*k2* = |2*k*2|

2*k2* = 2*k*2

The final step for this induction process would be to test the equation with the values n = k+1, or in this case x = k+1 for a constant value y, and y = k+1 for a constant value x.

Step 5:

2*y2* = |(*(k+1)+y*)(*(k+1)+*2*y*) – *(k+1)*(*(k+1)+*3*y*)|

2*y2*= |(*k*2+*k*+2*ky*+*k*+1+2*y*+*ky*+*y*+2*y*2) – (*k*2+*k*+3*ky*+*k*+1+3*y*)|

2*y2*= |(*k*2+2*k*+3*ky*+1+3*y*+2*y*2) – (*k*2+2*k*+3*ky*+1+3*y*)|

2*y2*= |*k*2+2*k*+3*ky*+1+3*y*+2*y*2 – *k*2+2*k*+3*ky*+1+3*y*|

2*y2*= |2*y*2|

2*y2* = 2*y*2

Step 6:

2*y2* = |(*(k+1)+y*)(*(k+1)+*2*y*) – *(k+1)*(*(k+1)+*3*y*)|

2*y2*= |(*k*2+*k*+2*ky*+*k*+1+2*y*+*ky*+*y*+2*y*2) – (*k*2+*k*+3*ky*+*k*+1+3*y*)|

2*y2*= |(*k*2+2*k*+3*ky*+1+3*y*+2*y*2) – (*k*2+2*k*+3*ky*+1+3*y*)|

2*y2*= |*k*2+2*k*+3*ky*+1+3*y*+2*y*2 – *k*2–2*k*–3*ky*–1–3*y*|

2*y2*= |2*y*2|

2*y2* = 2*y*2

By using the value k+1 for x and y, the final step of the induction conclusively proves that this theorem is in fact true for any real values of x and y.

**Conclusion**

Through the investigation the theorem has been proved two different ways, and it can be said with a large amount of certainty that it will hold true for any real numbers. One possible extension is to apply the theorem to complex numbers to see if it can be expanded from just real numbers. In addition, a variation of this theorem could be applied to a sequence with 6 numbers, or could be further generalized and used to find the difference between products of a sequence with n terms. There may also have been other ways of proving the theorem; however I believe that the two used are the simplest and most concise.

This theorem has potential applications in the areas of computer science, arts, economics and vector operations, amongst other areas.